Rutgers University: Algebra Written Qualifying Exam

August 2017: Problem 4 Solution

Exercise. Let G be a nontrivial finite group of prime power order, and let H be a normal subgroup of G. Show that H contains at least one non-identity element of the center of G.

Solution.

Suppose $|G| = p^k$, where p is prime and H is a normal subgroup of G.
Since $H \lhd G$, $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

If $H \lhd G$ then H is the union of the conjugacy classes of G.

$$H = [g_1] \cup [g_2] \cup \cdots \cup [g_n], \qquad \text{where } [g_i] \neq [g_j] \text{ for } i \neq j$$

Recall: $[a] = \{n \in G : g \in G \text{ s.t. } b = gag^{-1}\}$
$[e] = \{e\}$

Since H is a subgroup of G, $|H| \mid |G| = p^k$

$$\begin{aligned}
\implies |H| &\equiv 0 \mod p \text{ if } H \text{ is non-trivial.} \\
|H| &= |\{e\}| + |[g_1]| + \cdots + |[g_n]| \qquad \text{since conjugacy classes are either disjoint or identical.} \\
&= 1 + |[g_1]| + \cdots + |[g_n]| \\
\implies pm &= 1 + |[g_1]| + \cdots + |[g_n]| \text{ for some } m \in \mathbb{N}
\end{aligned}$$

But $|[g_i]|$ must also divide $|G| = p^k$, and the equation above shows that $|[g_i]|$ cannot be a multiple of p for every i.

$\implies \exists \ell$ s.t. $|[g_\ell]| = 1$ and $g_\ell \neq e$
$\implies g_\ell$ is in the center of G

Thus, H contains at least one non-identity element of the center of G, namely g_ℓ

Claim: $[a] = 1$ IFF a is in the center of G
Proof:

$$\begin{aligned}
[a] = 1 \quad &\iff \quad g^{-1}ag = a \quad \forall g \in G \\
&\iff \quad ag = ga \quad \forall g \in G \\
&\iff \quad a \text{ is in the center of } G
\end{aligned}$$